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Arithmetic Operations on Heptagonal Fuzzy Numbers

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper we have introduced Heptagonal fuzzy numbers (HpFNs) which deals with the membership function for the set of seven numbers. Arithmetic operations such as addition, subtraction, multiplication and division of two HpFNs based on $\alpha-$ cuts method, Zadeh's extension principle method and Co-ordinate method are discussed in detail. Membership functions are obtained for each method. Suitable examples and graphical illustrations are discussed for each method.

Keywords: Fuzzy set; Heptagonal fuzzy numbers (HpFNs); $\alpha-$ cuts.

1 Introduction

Many real time situations are not crisp and they are uncertain in nature, for example predicting the performance measure of the communication systems, mathematical modeling for the engineering problems, biological problems and computer systems. The parameter involved in such mathematical models is not precise as a result it leads to uncertainties and therefore it is vital to represent such uncertainties in linguistic characterization which can be attained through fuzzy numbers.

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The fuzzy set theory was first introduced by Zadeh [1] in 1965 as an extension of the classical notion of set which deals with degree of membership of elements in a set, elementary fuzzy calculus was initially developed by Goetschel & Voxman [2],Operations on fuzzy numbers was discussed by Dubois and Parade [3] then Atanassov generalized and introduced intuitionistic concepts in fuzzy environment [4-10] which deals with membership, non –membership and indeterminacy which has established intensifying attention since after introduction. The arithmetic operations on the intuitionistic fuzzy numbers discussed by Chang and Zadeh [11]. Wang [12] furnished the definition of intuitionistic trapezoidal fuzzy number and interval intuitionistic trapezoidal fuzzy number. Wang and Zhou [13] developed the intuitionistic trapezoidal fuzzy weighted arithmetic operator. Taleshian and Rezvani [14] discussed a Multiplication operation on trapezoidal fuzzy numbers, further the framework to calculate the membership degree of elements of a fuzzy set and functions discussed by Barros et al. [15]. New method of ranking generalized $L\text{-}R$ fuzzy numbers based on possibility theory discussed by Qiupeng G and Zuxing Xuan [16].Computational Methods for fuzzy arithmetic operations discussed by Akther and Ahamd [17]. Generalized operators for triangular intuitionistic fuzzy numbers was discussed by Wan, Wang, Ying Dong [18]. Further geometric aggregation operators with intuitionistic trapezoidal fuzzy numbers was discussed by Jian Wu, Qing-wei Cao [19]. The canonical representation on fuzzy numbers was introduced by Yong Deng [20]. An improved method to generalized fuzzy numbers with different left heights and right heights was discussed by Jiang Wen et al. [21]. Dong and Shah [22] introduced vertex method in fuzzy numbers by which the fuzzy variable can be calculated. Some operations on Intuitionistic pentagonal fuzzy number was discussed Ponnivalavan and Pathinathan [23], Arithmetic operators in interval-valued fuzzy set theory was discussed by Deschrijver [24].

Fuzzy tools are also widely applied mathematical modeling for engineering and medical fields .Puri and Ralescu [25] introduced Differential for fuzzy function. Bencsik, Bede, Tar and Fodor [26] applied Fuzzy differential equations in modeling hydraulic differential servo cylinders .Hassan Zarei et al. [27] applied Fuzzy Modeling in the Control of HIV Infection. Predator-Prey Model with Fuzzy Initial Populations was developed by Muhammad Zaini Ahmad, Bernard De Baets [28]. Buckley, Feuring, Hayashi [29] developed the solution to the first order ordinary differential equations using Fuzzy initial conditions. Seikkala [30] discussed on the fuzzy initial value problem.Garg [31,32] developed an approach for analysing the behaviour of industrial systems using weakest t-norm and intuitionistic fuzzy set theory and also for analysed the reliability of series-parallel system using credibility theory and different types of intuitionistic fuzzy numbers, Later Garg & Garg [33] discussed reliability Analysis of an Industrial System using T-Norm and T-Conorm Operations. Dhiman and Garg [34] discussed reliability Analysis of an Industrial System Using an Improved Arithmetic Operations. Rossell [35] introduced fuzzy concepts in biopolymers. Shapique [36] applied fuzzy concepts in control chart.

Recently new fuzzy numbers are formed and its arithmetic operations were discussed by various authors. A new and efficient method for elementary fuzzy arithmetic operations on pseudo-geometric fuzzy numbers was briefly discussed by Abbasi et al. [37]. Arithmetic operations on Pentagonal fuzzy numbers using $\alpha-$ cuts discussed by Bongu Lee and Yong Sik Yun [38],the Arithmetic operations on parabolic fuzzy numbers which was discussed by Garg and Ansha [39], Arithmetic operations on Octagonal Fuzzy Numbers by Malini [40], Arithmetic operations on Non-Newtonian by Kadak [41].

In this paper we have introduced Heptagonal fuzzy numbers (HpFNs) which will be useful to characterize the linguistic parameter. The commonly used fuzzy numbers (i.e) the triangular fuzzy have only three values one defining a single value estimate and other two represents widest possible interval for the parameter. The Heptagonal fuzzy numbers has seven numbers where the uncertainty involved in mathematical models can be characterized by Heptagonal fuzzy numbers. This paper has four sections, in the first section we have introduced the concept of fuzzy numbers, in the second section we have discussed preliminaries, notations of the fuzzy set and some properties related to Heptagon fuzzy numbers and in third section we have discussed the Arithmetic operations of two Heptagonal fuzzy numbers based on $\alpha-$ cuts, Zadeh's extension principle method and Co-ordinate method are discussed and in the last section we concluded our result.

2 Preliminaries and Notations

Definition 2.1. *Fuzzy set:*

A fuzzy set is characterized by a membership function mapping of a domain space (i.e) mapping between universe of discourse X to the unit interval [0,1] given by

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}(x) \right) / x \in X \right\}$$

Here $\mu_A : X \to [0,1]$ is called the degree of membership function of the fuzzy set A.

Definition 2.2. *Normal fuzzy set:*

A fuzzy set A of the universe of discourse X is called a normal fuzzy set if there exits at- least one $x \in X$ such that $\mu_A(x) = 1$.

Definition 2.3. *Height of a fuzzy set:*

The largest membership grade obtained by any element in the fuzzy set and it is given by $h(\tilde{A}) = \sup \ \mu_{\tilde{A}}(x)$

Definition 2.4. *Convex fuzzy set:*

A fuzzy set $\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}(x) \right) / x \in X \right\}$ is said to be convex if and only if for any $x_1, x_2 \in X$, the membership function of A satisfies the condition $\mu_{\tilde{A}}\left\{ \lambda x_1 + (1-\lambda) x_2 \right\} \geq \min\left\{ \mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2) \right\}, \lambda \in [0,1]$

Definition 2.5. *Extension principle*:

Consider a mapping f is defined on the power set of the universe $f : P\left(X_1 \text{ x } X_2 \text{ x} \ldots\ldots \text{x } X_n \right) \to P(Y)$, let the fuzzy sets $A_1, A_2 \ldots\ldots A_n$ be defined on $X_1, X_2, \ldots\ldots, X_n$ and $B = f\left(A_1, A_2 \ldots\ldots A_n \right)$ then the membership function B is defined as

$$\mu_B(y) = \sup_{y = f(x_1, x_2 \ldots\ldots x_n)} \left\{ \min\left[\mu_{A_1}(x_1), \mu_{A_2}(x_2) \ldots\ldots \mu_{A_n}(x_n) \right] \right\}$$

Definition 2.6. *Heptagon Fuzzy numbers:*

A HpFNs $\tilde{A}_{Hp} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ *is a subset of fuzzy number in R with following membership function*

$$\mu_{\tilde{A}_{Hp}}(x) = \begin{cases} \dfrac{x-a_1}{2(a_2-a_1)}, & a_1 \le x \le a_2 \\ \dfrac{1}{2}, & a_2 \le x < a_3 \\ \dfrac{x-a_4}{2(a_4-a_3)}+1, & a_3 \le x \le a_4 \\ \dfrac{a_4-x}{2(a_5-a_4)}+1, & a_4 \le x \le a_5 \\ \dfrac{1}{2}, & a_5 \le x < a_6 \\ \dfrac{a_7-x}{2(a_7-a_6)}, & a_6 \le x \le a_7 \\ 0, & \text{otherwise} \end{cases}$$

Where $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7$ *and HpFN is denoted by* $\tilde{A}_{Hp} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$

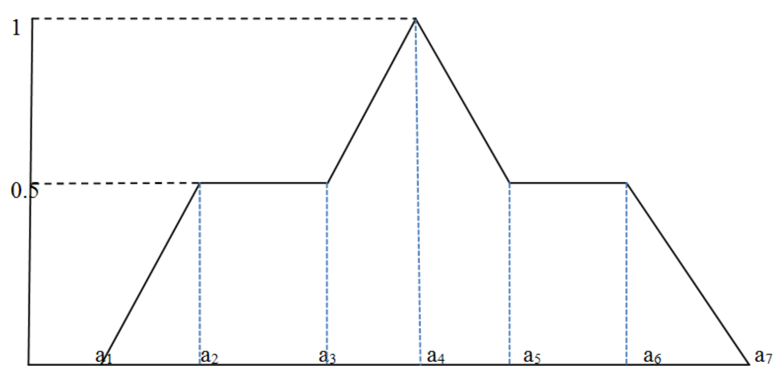

Fig. 1. Graphical representation of the HpFNs

Property 2.7

a. The HpFNs $\tilde{A}_{Hp} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ is said to symmetric if the product of the difference of two set (a_1, a_2, a_3, a_4) and (a_4, a_5, a_6, a_7) are equal. (i.e) $|a_1 - a_2|.|a_2 - a_3|.|a_3 - a_4| = |a_5 - a_6|.|a_6 - a_7|.|a_7 - a_8|$

Example $\tilde{A}_{Hp} = (2,3,4,5,6,7,8)$

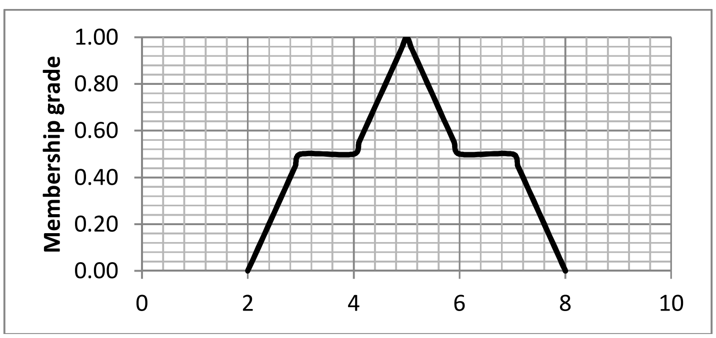

Fig. 2. Graphical representation of symmetric HpFNs

b. The HpFNs $\tilde{A}_{Hp} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ *is said to non-symmetric or skewed right if* $|a_1 - a_2|.|a_2 - a_3|.|a_3 - a_4| > |a_5 - a_6|.|a_6 - a_7|.|a_7 - a_8|$

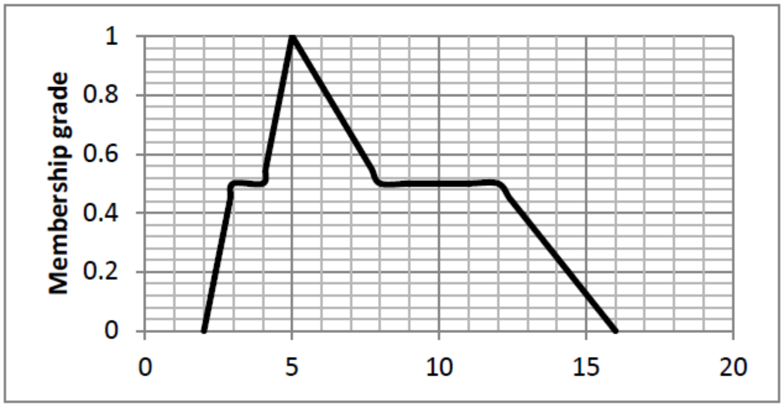

Fig. 3. Graphical representation of non-symmetric HpFNs

c. The HpFNs $\tilde{A}_{Hp} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ *is said to non-symmetric or skewed left if* $|a_1 - a_2|.|a_2 - a_3|.|a_3 - a_4| < |a_5 - a_6|.|a_6 - a_7|.|a_7 - a_8|$

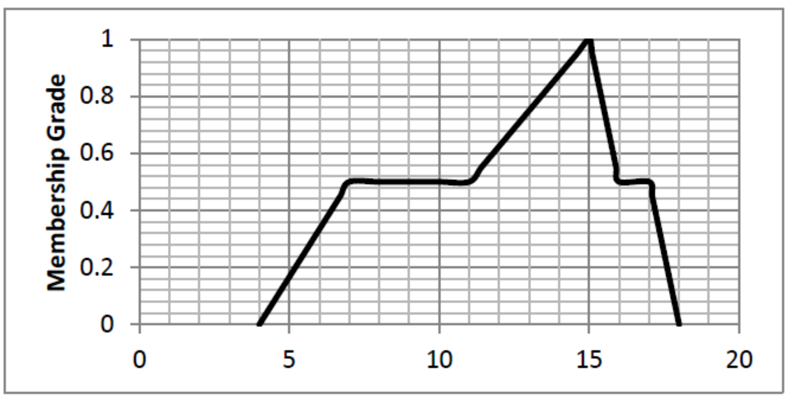

Fig. 4. Graphical representation of non-symmetric HpFNs

3 Arithmetic Operations of Two HpFNs

3.1 Arithmetic operations based on $\alpha-$cuts

Let's consider two HpFNs $\tilde{A}_{Hp} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ and $\tilde{B}_{Hp} = (\mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3, \mathrm{b}_4, \mathrm{b}_5, \mathrm{b}_6, \mathrm{b}_7)$ whose membership function is given as follows

$$\mu_{\tilde{A}_{Hp}}(x)=\begin{cases}\frac{x-a_1}{2(a_2-a_1)}, & a_1\le x\le a_2\\ \frac{1}{2}, & a_2\le x<a_3\\ \frac{x-a_4}{2(a_4-a_3)}+1, & a_3\le x\le a_4\\ \frac{a_4-x}{2(a_5-a_4)}+1, & a_4\le x\le a_5\\ \frac{1}{2}, & a_5\le x<a_6\\ \frac{a_7-x}{2(a_7-a_6)}, & a_6\le x\le a_7\\ 0, & otherwise\end{cases} \quad \text{and} \quad \mu_{\tilde{B}_{Hp}}(x)=\begin{cases}\frac{x-b_1}{2(b_2-b_1)}, & b_1\le x\le b_2\\ \frac{1}{2}, & b_2\le x<b_3\\ \frac{x-b_4}{2(b_4-b_3)}+1, & b_3\le x\le b_4\\ \frac{b_4-x}{2(b_5-b_4)}+1, & b_4\le x\le b_5\\ \frac{1}{2}, & b_5\le x<b_6\\ \frac{b_7-x}{2(b_7-b_6)}, & b_6\le x\le b_7\\ 0, & otherwise\end{cases}$$

If $\tilde{A}_{Hp}$ and $\tilde{B}_{Hp}$ are any HpFN, then their $\alpha-$ cuts are given by

$$[A_\alpha]=\begin{cases}[A_1(\alpha),A_2(\alpha))] & for\ \alpha\in[0,0.5]\\ [A_3(\alpha),A_4(\alpha)] & for\ \alpha\in[0.5,1]\end{cases}$$

$$[B_\alpha]=\begin{cases}[B_1(\alpha),B_2(\alpha))] & for\ \alpha\in[0,0.5]\\ [B_3(\alpha),B_4(\alpha)] & for\ \alpha\in[0.5,1]\end{cases}$$

$$A_\alpha=\begin{cases}a_1+2\alpha(a_2-a_1),a_7-2\alpha(a_7-a_6) & ,\alpha\in[0,0.5]\\ 2(\alpha-1)(a_4-a_3)+a_4,a_4-2(\alpha-1)(a_5-a_4)\ , & \alpha\in[0.5,1]\end{cases}$$

$$B_\alpha=\begin{cases}b_1+2\alpha(b_2-b_1),b_7-2\alpha(b_7-b_6) & ,\alpha\in[0,0.5]\\ 2(\alpha-1)(b_4-b_3)+b_4,b_4-2(\alpha-1)(b_5-b_4) & ,\alpha\in[0.5,1]\end{cases}$$

3.2 Sum of two HpFNs by $(\alpha)-$ cut method

For $0\le\alpha\le0.5$

$$A_\alpha\oplus B_\alpha=\left[A_1^{\ \alpha}+B_1^{\ \alpha},A_2^{\ \alpha}+B_2^{\ \alpha}\right]$$

$$=[a_1+b_1+2\alpha(a_2+b_2-a_1-b_1),a_7+b_7-2\alpha(a_7+b_7-a_6-b_6)]$$

If $x\in[a_1+b_1,a_2+b_2]$ then $x=a_1+b_1+2\alpha(a_2+b_2-a_1-b_1)$

Thus $\alpha=\dfrac{x-a_1-b_1}{2(a_2+b_2-a_1-b_1)}\quad for\ \ 0\le\alpha\le0.5$

If $x \in [a_6 + b_6, a_7 + b_7]$ *then* $x = a_7 + b_7 - 2\alpha(a_7 + b_7 - a_6 - b_6)$ *thus* $\alpha = \dfrac{x - a_7 - b_7}{2(a_6 + b_6 - a_7 - b_7)}$

For $0.5 \le \alpha \le 1$, $A_\alpha \oplus B_\alpha = [A_1^\alpha + B_1^\alpha, A_2^\alpha + B_2^\alpha]$

$$= [a_4 + b_4 + 2(\alpha - 1)(a_4 + b_4 - a_3 - b_3), a_4 + b_4 - 2(\alpha - 1)(a_5 + b_5 - a_4 - b_4)]$$

If $x \in [a_3 + b_3, a_4 + b_4]$ *then* $x = a_4 + b_4 + 2(\alpha - 1)(a_4 + b_4 - a_3 - b_3)$

Thus $\alpha = \dfrac{x - a_4 - b_4}{2(a_4 + b_4 - a_3 - b_3)} + 1$, *for* $0.5 \le \alpha \le 1$

If $x \in [a_4 + b_4, a_5 + b_5]$ $x = a_4 + b_4 - 2(\alpha - 1)(a_5 + b_5 - a_4 - b_4)$

Thus $\alpha = \dfrac{x - a_4 - b_4}{2(a_4 + b_4 - a_5 - b_5)} + 1$, therefore the membership function is given by

$$\mu_{\tilde{A}_{Hp} \oplus \tilde{B}_{Hp}}(x) = \begin{cases} \dfrac{x - a_1 - b_1}{2(a_2 + b_2 - a_1 - b_1)}, & a_1 + b_1 \le x \le a_2 + b_2 \\ \dfrac{1}{2}, & a_2 + b_2 \le x < a_3 + b_3 \\ \dfrac{x - a_4 - b_4}{2(a_4 + b_4 - a_3 - b_3)} + 1, & a_3 + b_3 \le x \le a_4 + b_4 \\ \dfrac{a_4 + b_4 - x}{2(a_5 + b_5 - a_4 - b_4)} + 1, & a_4 + b_4 \le x \le a_5 + b_5 \\ \dfrac{1}{2}, & a_5 + b_5 \le x \le a_6 + b_6 \\ \dfrac{a_7 + b_7 - x}{2(a_7 + b_7 - a_6 - b_6)}, & a_6 + b_6 \le x \le a_7 + b_7 \\ 0, & x < a_1 + b_1, a_7 + b_7 \le x \end{cases} \quad (3.2.1)$$

Thus If $\tilde{A}_{Hp} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ and $\tilde{B}_{Hp} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$ are any two HpFS then $\tilde{A}_{Hp} \oplus \tilde{B}_{Hp}$ is also HpFS and it is given by $\tilde{A}_{Hp} \oplus \tilde{B}_{Hp} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6, a_7 + b_7)$

Example:

Let $\tilde{A}_{Hp} = (1,3,5,7,9,11,13)$ $\tilde{B}_{Hp} = (2,4,6,8,10,12,14)$ then the graph of the membership function $\tilde{A}_{Hp} \oplus \tilde{B}_{Hp}$ is given as follows

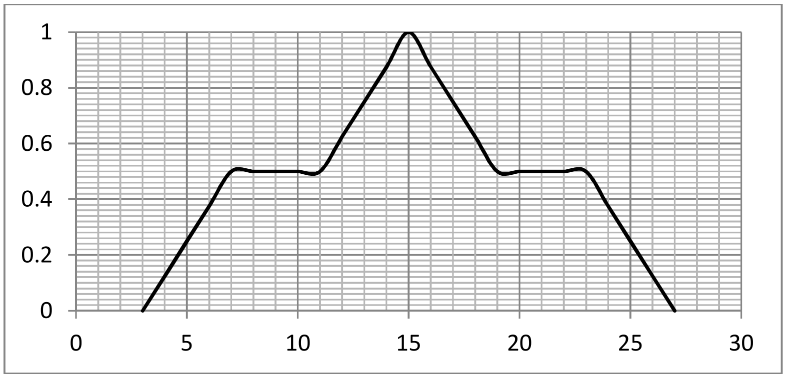

Fig. 5. Graphical representation of the membership function of sum of two the HpFNs

3.3 Sum of two HpFNs by extension principle method

Let $\tilde{A}_{Hp} = (a_1, a_2, a_3, \mathrm{a}_4, \mathrm{a}_5, \mathrm{a}_6, \mathrm{a}_7)$ and $\tilde{B}_{Hp} = (\mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3, \mathrm{b}_4, \mathrm{b}_5, \mathrm{b}_6, \mathrm{b}_7)$ are any two HpFNs with the membership functions

$$\mu_{\tilde{A}}(\mathrm{x}) = \sup\left(\min\left(\frac{x-a_1}{2(a_2-a_1)}, \frac{1}{2}, \frac{x-a_4}{2(a_4-a_3)}+1, \frac{a_4-x}{2(a_5-a_4)}+1, \frac{a_7-x}{2(a_7-a_6)} \right), 0 \right)$$

$$\mu_{\tilde{B}}(\mathrm{y}) = \sup\left(\min\left(\frac{y-b_1}{2(b_2-b_1)}, \frac{1}{2}, \frac{y-b_4}{2(\mathrm{b}_4-b_3)}+1, \frac{b_4-y}{2(\mathrm{b}_5-b_4)}+1, \frac{b_7-y}{2(b_7-b_6)} \right), 0 \right)$$

Let $\tilde{A} \oplus \tilde{B} = \tilde{C}, \mu_{\tilde{c}}(\mathrm{z}) = \sup(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(\mathrm{y})) / x+y=z)$

For $0 \le \alpha \le 0.5$

$$\mu_{\tilde{c}}(z) = \begin{cases} \mu^{L}{}_{\tilde{c}}(\mathrm{z}) = \sup\left(\min\left(\frac{x-a_1}{2(a_2-a_1)}, \frac{y-b_1}{2(b_2-b_1)} \right) / x+y=z \right) \text{if } a_1 < x < a_2, b_1 < y < b_2 \\ \sup\left(\min\left(\frac{1}{2}, \frac{1}{2} \right) / \mathrm{x}+\mathrm{y}=\mathrm{z} \right) \text{ if } a_2 < x < a_3, b_2 < y < b_3, a_5 < x < a_6, b_5 < y < b_6 \\ \mu^{R}{}_{\tilde{c}}(\mathrm{z}) = \sup\left(\min\left(\frac{a_7-x}{2(a_7-a_6)}, \frac{b_7-y}{2(b_7-b_6)} \right) / x+y=z \right) \text{if } a_6 < x < a_7, b_6 < y < b_7 \\ 0, \text{otherwise} \end{cases}$$

If we choose $\alpha = \min\left(\frac{x-a_1}{2(a_2-a_1)}, \frac{y-b_1}{2(b_2-b_1)} \right)$

Then $\alpha \le \frac{x-a_1}{2(a_2-a_1)}, \alpha \le \frac{y-b_1}{2(b_2-b_1)}$ *therefore* $\alpha \le \frac{x+y-a_1-b_1}{2(a_2+b_2-a_1-b_1)}$

$$\mu^{L}{}_{\tilde{c}}(z) = \sup\alpha = \frac{x+y-a_1-b_1}{2(a_2+b_2-a_1-b_1)}$$

$$\mu^{L}{}_{\tilde{c}}(z) = \frac{z-a_1-b_1}{2(a_2+b_2-a_1-b_1)}$$

Similarly $\mu^{R}{}_{\tilde{c}}(z) = \dfrac{a_7+b_7-z}{2(a_7+b_7-a_6-b_6)}$

$For\ 0.5 < \alpha \le 1$

$$\mu_{\tilde{c}}(z) = \begin{cases} \mu^{L}{}_{\tilde{c}}(z) = \sup\left(\min\left(\dfrac{x-a_4}{2(a_4-a_3)}+1, \dfrac{y-b_4}{2(b_4-b_3)}+1\right)/\, x+y=z\right) if\ a_3 < x < a_4, b_3 < y < b_4 \\ \mu^{R}{}_{\tilde{c}}(z) = \sup\left(\min\left(\dfrac{a_4-x}{2(a_5-a_4)}+1, \dfrac{b_4-y}{2(b_5-b_4)}+1\right)/\, x+y=z\right) if\ a_4 < x < a_5, b_4 < y < b_5 \\ 0, Otherwise \end{cases}$$

If we choose $\alpha = \min\left(\dfrac{x-a_4}{2(a_4-a_3)}+1, \dfrac{y-b_4}{2(b_4-b_3)}+1\right)$

Then $\alpha \le \dfrac{x-a_4}{2(a_4-a_3)}+1, \alpha \le \dfrac{y-b_4}{2(b_4-b_3)}+1$ *therefore* $\alpha \le \dfrac{x+y-a_4-b_4}{2(a_4+b_4-a_3-b_3)}+1$

$$\sup\alpha = \frac{x+y-a_4-b_4}{2(a_4+b_4-a_3-b_3)}+1$$

$\mu^{L}{}_{\tilde{c}}(z) = \dfrac{z-a_4-b_4}{2(a_4+b_4-a_3-b_3)}+1$ Similarly we can obtain $\mu^{R}{}_{\tilde{c}}(z) = \dfrac{a_4+b_4-z}{2(a_5+b_5-a_4-b_4)}+1$

Hence the membership function obtained is same as (3.2.1).

3.4 Sum of two HpFN using co-ordinates method

We define $\tilde{A} \oplus \tilde{B} = \tilde{C} = \phi(\tilde{A}, \tilde{B})$

$consider,\ for\ 0 \le \alpha \le 0.5$

$$x_1 = a_1 - 2\alpha(a_2 - a_1), y_1 = \mathrm{b}_1 - 2\alpha(\mathrm{b}_2 - b_1)$$

$$x_2 = a_7 - 2\alpha(a_7 - a_6), y_2 = \mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)$$

Now we form the coordinate of the vertices

$$c_1 = (x_1, y_1), c_2 = (x_1, y_2) \quad c_3 = (x_2, y_1), c_4 = (x_2, y_2)$$

$$\phi(c_1) = (x_1 + y_1), \phi(c_2) = (x_1 + y_2), \phi(c_3) = (x_2 + y_1), \phi(c_4) = (x_2 + y_2)$$

$$z = [\min \phi(c_1), \phi(c_2), \phi(c_3), \phi(c_4), \max \phi(c_1), \phi(c_2), \phi(c_3), \phi(c_4)]$$

$$= [\phi(c_1), \phi(c_4)]$$

$$z = [a_1 + b_1 + 2\alpha(a_2 + b_2 - a_1 - b_1), a_7 + b_7 - 2\alpha(a_7 + b_7 - a_6 - b_6)]$$

Collecting α we get

$$\mu^L{}_{\tilde{c}}(z) = \frac{z - a_1 - b_1}{2(a_2 + b_2 - a_1 - b_1)} \quad \mu^R{}_{\tilde{c}}(z) = \frac{a_7 + b_7 - z}{2(a_7 + b_7 - a_6 - b_6)}$$

Similarly, *consider for* $0.5 < \alpha \le 1$

Where $x_3 = 2(\alpha - 1)(a_4 - a_3) + a_4, y_3 = 2(\alpha - 1)(b_4 - b_3) + b_4$

$$x_4 = a_4 - 2(\alpha - 1)(a_5 - a_4), y_4 = b_4 - 2(\alpha - 1)(b_5 - b_4)$$

$$c_5 = (x_3, y_3), c_6 = (x_3, y_4), c_7 = (x_4, y_3), c_8 = (x_4, y_4)$$

Now $\varphi(c_5) = (x_3 + y_3), \varphi(c_6) = (x_3 + y_4), \varphi(c_7) = (x_4 + y_3), \varphi(c_8) = (x_4 + y_4)$

$$z = [\min \varphi(c_5), \varphi(c_6), \varphi(c_7), \varphi(c_8), \max \varphi(c_5), \varphi(c_6), \varphi(c_7), \varphi(c_8)]$$

$$= [\varphi(c_5), \varphi(c_8)]$$

$$z = [a_4 + b_4 + 2(\alpha - 1)(a_4 - a_3 + b_4 - b_3), a_4 + b_4 - 2(\alpha - 1)(a_5 - a_4 + b_5 - b_4)]$$

Collecting α we get

$$\mu^L{}_{\tilde{c}}(z) = \frac{z - a_4 - b_4}{2(a_4 + b_4 - a_3 - b_3)} + 1, \mu^R{}_{\tilde{c}}(z) = \frac{a_4 + b_4 - z}{2(a_5 + b_5 - a_4 - b_4)} + 1$$

Hence the membership function obtained is same as (3.2.1).

3.5 Subtraction of two HpFNs by (α) – cut method

For $0 \le \alpha \le 0.5$

$$A_\alpha(-)B_\alpha = \left[A_1^\alpha - B_2^\alpha, A_2^\alpha - B_1^\alpha \right]$$

$$= [a_1 - b_7 + 2\alpha(a_2 + b_7 - a_1 - b_6), a_7 - b_1 - 2\alpha(a_7 + b_2 - a_6 - b_1)]$$

If $x \in [a_1 - b_7, a_2 - b_6]$ then $x = a_1 - b_7 + 2\alpha(a_2 + b_7 - a_1 - b_6)$

Thus $\alpha = \dfrac{x - a_1 + b_7}{2(a_2 + b_7 - a_1 - b_6)}$ *for* $0 \le \alpha \le 0.5$

If $x \in [a_6 - b_2, a_7 - b_1]$ *then* $x = a_7 - b_1 - 2\alpha(a_7 + b_2 - a_6 - b_1)$ $\alpha = \dfrac{x - a_7 + b_1}{2(a_6 + b_1 - a_7 - b_2)}$

For $0.5 < \alpha \le 1$

$$A_\alpha(-)B_\alpha = \left[A_1^{\alpha} - B_2^{\alpha}, A_2^{\alpha} - B_1^{\alpha}\right]$$

$$= [a_4 - b_4 + 2(\alpha - 1)(a_4 + b_5 - a_3 - b_4), a_4 - b_4 - 2(\alpha - 1)(a_5 + b_4 - a_4 - b_3)]$$

If $x \in [a_3 - b_5, a_4 - b_4]$ *then* $x = a_4 - b_4 + 2(\alpha - 1)(a_4 + b_5 - a_3 - b_4)$

Thus $\alpha = \dfrac{x - a_4 + b_4}{2(a_4 + b_5 - a_3 - b_4)} + 1$

Similarly If $x \in [a_4 - b_4, a_5 - b_3]$ *then* $x = a_4 - b_4 - 2(\alpha - 1)(a_5 + b_4 - a_4 - b_3)$

$$\alpha = \frac{x - a_4 + b_4}{2(a_4 + b_3 - a_5 - b_4)} + 1, \textit{ for } 0.5 \le \alpha \le 1$$

Therefore the membership function is given by

$$\mu_{\tilde{A}_{Hp}(-)\tilde{B}_{Hp}}(x) = \begin{cases} \dfrac{x - a_1 + b_7}{2(a_2 + b_7 - a_1 - b_6)}, & a_1 - b_7 \le x \le a_2 - b_6 \\ \dfrac{1}{2}, & a_2 - b_6 \le x < a_3 - b_5 \\ \dfrac{x - a_4 + b_4}{2(a_4 + b_5 - a_3 - b_4)} + 1, & a_3 - b_5 \le x \le a_4 - b_4 \\ \dfrac{x - a_4 + b_4}{2(a_4 + b_3 - a_5 - b_4)} + 1, & a_4 - b_4 \le x \le a_5 - b_3 \\ \dfrac{1}{2}, & a_5 - b_3 \le x \le a_6 - b_2 \\ \dfrac{x - a_7 + b_1}{2(a_6 + b_1 - a_7 - b_2)}, & a_6 - b_2 \le x \le a_7 - b_1 \\ 0, & x < a_1 - b_7, a_7 - b_1 \le x \end{cases} \quad (3.5.1)$$

Thus If $\tilde{A}_{Hp} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ and $\tilde{B}_{Hp} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$ are any two HpFS then $\tilde{A}_{Hp}(-)\tilde{B}_{Hp}$ is also HpFS and it is given *by* $\tilde{A}_{Hp}(-)\tilde{B}_{Hp} = (a_1 - b_7, a_2 - b_6, a_3 - b_5, a_4 - b_4, a_5 - b_3, a_6 - b_2, a_7 - b_1)$

Example:

Let $\tilde{A}_{Hp} = (1,4,5,9,11,14,15)$ and $\tilde{B}_{Hp} = (2,3,7,10,12,16,18)$ be ant two HpFns then the graph of the membership function $\tilde{A}_{Hp}(-)\tilde{B}_{Hp}$ is given as follows

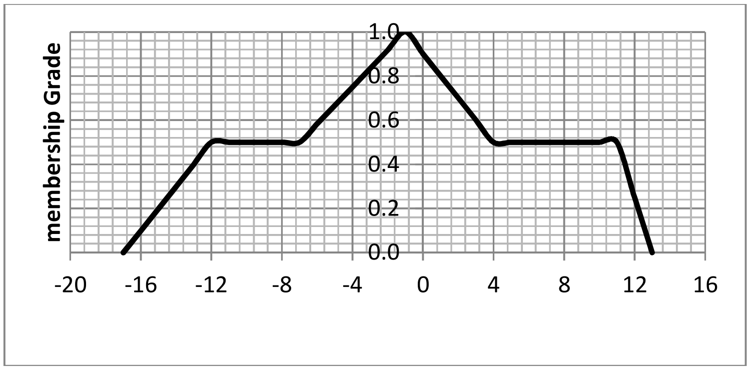

Fig. 6. Graphical representation of the membership function of subtraction of two the HpFNs

3.6 Subtraction of two HpFNS by extension principle method

Let $\tilde{A}(-)\tilde{B} = \tilde{C}$ where $\mu_{\tilde{c}}(z) = \sup(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(\text{y})) / x - y = z)$

For $0 < \alpha \le 0.5$

$$\mu_{\tilde{c}}(z) = \begin{cases} \mu^{L}{}_{\tilde{c}}(z) = \sup\left(\min\left(\dfrac{x-a_1}{2(a_2-a_1)}, \dfrac{y-b_7}{2(b_6-b_7)}\right) / x-y=z\right) if\ a_1 \le x \le a_2, b_6 \le y \le b_7 \\ \sup\left(\min\left(\dfrac{1}{2}, \dfrac{1}{2}\right) / x-y=z\right) if\ a_2 \le x \le a_3, b_2 \le y \le b_3, a_5 \le x \le a_6, b_5 \le y \le b_6 \\ \mu^{R}{}_{\tilde{c}}(z) = \sup\left(\min\left(\dfrac{a_7-x}{2(a_7-a_6)}, \dfrac{x-b_1}{2(\text{b}_2-b_1)}\right) / x-y=z\right) if\ a_6 < x < a_7, b_1 < y < b_2 \\ 0, \text{otherwise} \end{cases}$$

If we choose $\min\left(\dfrac{x-a_1}{2(a_2-a_1)}, \dfrac{y-b_7}{2(b_6-b_7)}\right) = \alpha$

$$\alpha \le \frac{x-a_1}{2(a_2-a_1)}, \alpha \le \frac{y-b_7}{2(b_6-b_7)}\ therefore\ \alpha \le \frac{x-y-a_1+b_7}{2(a_2-b_6-a_1+b_7)}$$

$$\mu^{L}{}_{\tilde{c}}(z) = \frac{z-a_1+b_7}{2(a_2-b_6-a_1+b_7)}$$

Also by applying same technique we find $\mu^{R}{}_{\tilde{c}}(z) = \dfrac{x-a_7+b_1}{2(a_6+b_1-a_7-\text{b}_2)}$

Similarly *for* $0.5 < \alpha \le 1$

$$\mu_{\tilde{c}}(z) = \begin{cases} \mu^{L}{}_{\tilde{c}}(z) = \sup\left(\min\left(\dfrac{x-a_4}{2(a_4-a_3)}+1, \dfrac{b_4-y}{2(b_5-b_4)}+1 \right) / \, x-y=z \right) if \; a_3 \le x \le a_4, b_4 \le y \le b_5 \\ \mu^{R}{}_{\tilde{c}}(z) = \sup\left(\min\left(\dfrac{a_4-x}{2(a_5-a_4)}+1, \dfrac{y-b_4}{2(b_4-b_3)}+1 \right) / \, x-y=z \right) if \; a_4 \le x \le a_5, b_3 \le y \le b_4 \\ 0, \text{otherwise} \end{cases}$$

By using above procedure we find

$$\mu^{L}{}_{\tilde{c}}(z) = \frac{z-a_4+b_4}{2(a_4+b_5-a_3-\mathrm{b}_4)}+1 \text{ and } \mu^{R}{}_{\tilde{c}}(z) = \frac{z-a_4+b_4}{2(a_4+b_3-a_5-\mathrm{b}_4)}+1$$

Hence the membership function obtained is same as (3.5.1).

3.7 Subtraction of HpFNs using co-ordinates method

We define $\tilde{A}(-)\tilde{B} = \tilde{C} = \phi(\tilde{A}, \tilde{B})$

consider $x_1 = a_1 - 2\alpha(a_2 - a_1), y_1 = \mathrm{b}_1 - 2\alpha(\mathrm{b}_2 - b_1)$, $x_2 = a_7 - 2\alpha(a_7 - a_6), y_2 = \mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)$

Now we form combinations

$$c_1 = (x_1, y_1), c_2 = (x_1, y_2) \quad c_3 = (x_2, y_1), c_4 = (x_2, y_2)$$

$$\phi(c_1) = (x_1 - y_1), \phi(c_2) = (x_1 - y_2), \phi(c_3) = (x_2 - y_1), \phi(c_4) = (x_2 - y_2)$$

$$z = [\min \phi(c_1), \phi(c_2), \phi(c_3), \phi(c_4), \max \phi(c_1), \phi(c_2), \phi(c_3), \phi(c_4)]$$

$$z = [\phi(c_2), \phi(c_3)]$$

$$z = [a_1 - \mathrm{b}_7 - 2\alpha(a_2 + \mathrm{b}_6 - a_1 - b_7), a_7 - \mathrm{b}_1 - 2\alpha(a_7 + \mathrm{b}_1 - a_6 - b_2)]$$

Collecting α we get

$$\mu^{L}{}_{\tilde{c}}(z) = \frac{z-a_1+b_7}{2(a_2-b_6-a_1+b_7)} \text{ and } \mu^{R}{}_{\tilde{c}}(z) = \frac{x-a_7+b_1}{2(a_6+b_1-a_7-\mathrm{b}_2)}$$

Similarly we can find

$$z = [a_4 - b_4 + (2\alpha-1)(a_4 + b_5 - a_3 - b_4), a_4 - b_4 - (2\alpha-1)(a_5 + b_4 - a_4 - b_3)]$$

$$\mu^{L}{}_{\tilde{c}}(z) = \frac{z-a_4+b_4}{2(a_4+b_5-a_3-\mathrm{b}_4)}+1 \; \& \; \mu^{R}{}_{\tilde{c}}(z) = \frac{z-a_4+b_4}{2(a_4+b_3-a_5-\mathrm{b}_4)}+1$$

Hence we notice that subtraction of two HpFN is again a HpFN.

3.8 Product of two HpFNs by $(\alpha)-$ cut method

For $0 \le \alpha \le 0.5$

$$A_\alpha \otimes B_\alpha = \left[A_1^\alpha B_1^\alpha , A_2^\alpha B_2^\alpha \right]$$

$$= \left[\{a_1 + 2\alpha(a_2 - a_1)\}\{b_1 + 2\alpha(\mathrm{b}_2 - b_1)\}, \{a_7 - 2\alpha(a_7 - a_6)\}\{\mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)\} \right]$$

If $x \in [a_1b_1, a_2b_2]$ then $x = \{a_1 + 2\alpha(a_2 - a_1)\}\{b_1 + 2\alpha(\mathrm{b}_2 - b_1)\}$

Thus $\alpha = \dfrac{2a_1b_1 - a_2b_1 - a_1b_2 \pm \sqrt{f_1(x)}}{4\left(a_1b_1 - a_2b_1 - a_1b_2 + a_2b_2\right)}$

If $x \in [a_6 + b_6, a_7 + b_7]$ then $\mathrm{x} = \{a_7 - 2\alpha(a_7 - a_6)\}\{\mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)\}$

Thus $\alpha = \dfrac{2a_7b_7 - a_7b_6 - a_6b_7 \pm \sqrt{f_4(x)}}{4\left(a_6b_6 - a_7b_6 - a_6b_7 + a_7b_7\right)}$

Where. $f_1(x) = \left(a_2b_1\right)^2 - 2a_1a_2b_1b_2 + \left(a_1b_2\right)^2 + 4(a_1b_1 - a_2b_1 - a_1b_2 + a_2b_2)x$.

$$f_4(x) = \left(a_7b_6\right)^2 - 2a_6a_7b_6b_7 + \left(a_6b_7\right)^2 + 4(a_6b_6 - a_7b_6 - a_6b_7 + a_7b_7)x$$

For $0.5 \le \alpha \le 1$

$$A_\alpha \otimes B_\alpha = \left[A_2^\alpha B_2^\alpha , A_3^\alpha B_3^\alpha \right]$$

$$= \left[\{2(\alpha - 1)(a_4 - a_3) + \mathrm{a}_4\}\{2(\alpha - 1)(\mathrm{b}_4 - b_3) + b_4\}, \{\mathrm{a}_4 - 2(\alpha - 1)(a_5 - a_4)\}\{\mathrm{b}_4 - 2(\alpha - 1)(\mathrm{b}_5 - b_4)\} \right]$$

If $x \in [a_3 + b_3, a_4 + b_4]$ then $x = \{2(\alpha - 1)(a_4 - a_3) + \mathrm{a}_4\}\{2(\alpha - 1)(\mathrm{b}_4 - b_3) + b_4\}$

Thus $\alpha = \dfrac{4a_3b_3 - 3a_4b_3 - 3a_3b_4 + 2a_4b_4 \pm \sqrt{f_2(x)}}{4\left(a_3b_3 - a_4b_3 - a_3b_4 + a_4b_4\right)}$

Similarly If $x \in [a_4 + b_4, a_5 + b_5]$ then $\mathrm{x} = \{\mathrm{a}_4 - 2(\alpha - 1)(a_5 - a_4)\}\{\mathrm{b}_4 - 2(\alpha - 1)(\mathrm{b}_5 - b_4)\}$

$$\alpha = \frac{2a_4b_4 - 3a_5b_4 - 3a_4b_5 + 4a_5b_5 \pm \sqrt{f_3(x)}}{4\left(a_4b_4 - a_5b_4 - a_4b_5 + a_5b_5\right)}$$

Where

$$f_2(x) = \left(a_4b_3\right)^2 - 2a_3a_4b_3b_4 + \left(a_3b_4\right)^2 + 4(a_3b_3 - a_4b_3 - a_3b_4 + a_4b_4)x$$

$$f_3(x) = (a_5b_4)^2 - 2a_4a_5b_4b_5 + (a_4b_5)^2 + 4(a_4b_4 - a_5b_4 - a_4b_5 + a_5b_5)x$$

Therefore

$$\mu_{\tilde{A}_{Hp}\otimes\tilde{B}_{Hp}}(x) = \begin{cases} \dfrac{2a_1b_1 - a_2b_1 - a_1b_2 + \sqrt{f_1(x)}}{4(a_1b_1 - a_2b_1 - a_1b_2 + a_2b_2)}, & a_1b_1 \le x \le a_2b_2 \\ \dfrac{1}{2}, & a_2b_2 \le x < a_3b_3 \\ \dfrac{4a_3b_3 - 3a_4b_3 - 3a_3b_4 + 2a_4b_4 + \sqrt{f_2(x)}}{4(a_3b_3 - a_4b_3 - a_3b_4 + a_4b_4)}, & a_3b_3 \le x \le a_4b_4 \\ \dfrac{2a_4b_4 - 3a_5b_4 - 3a_4b_5 + 4a_5b_5 - \sqrt{f_3(x)}}{4(a_4b_4 - a_5b_4 - a_4b_5 + a_5b_5)}, & a_4b_4 \le x \le a_5b_5 \\ \dfrac{1}{2}, & a_5b_5 \le x \le a_6b_6 \\ \dfrac{2a_7b_7 - a_7b_6 - a_6b_7 - \sqrt{f_4(x)}}{4(a_6b_6 - a_7b_6 - a_6b_7 + a_7b_7)}, & a_6b_6 \le x \le a_7b_7 \\ 0, & otherwise \end{cases} \qquad (3.8.1)$$

Thus If $\tilde{A}_{Hp} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ and $\tilde{B}_{Hp} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$ are any two HpFS then $\tilde{A}_{Hp} \otimes \tilde{B}_{Hp}$ is also HpFS and it is given by $\tilde{A}_{Hp} \otimes \tilde{B}_{Hp} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6, a_7b_7)$

Example: Let $\tilde{A}_{Hp} = (2,4,6,8,10,12,14)$ $\tilde{B}_{Hp} = (1,3,5,7,9,11,13)$ then the graph of the membership function $\tilde{A}_{Hp} \otimes \tilde{B}_{Hp}$ is given as follows

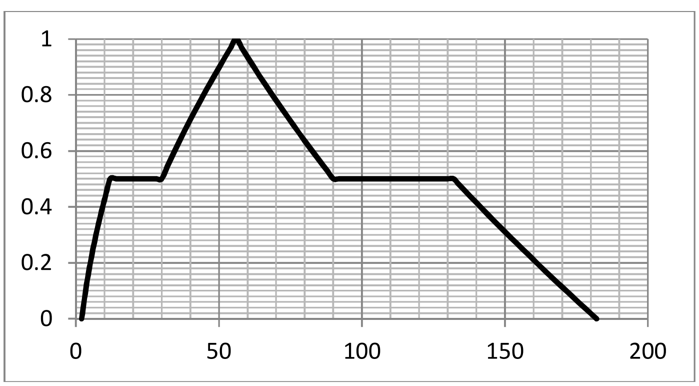

Fig. 7. Graphical representation of the membership function of product of two the HpFNs

3.9 Product of two HpFN using extension principle method:

Let $\tilde{A}\otimes\tilde{B}=\tilde{C}$ where $\mu_{\tilde{c}}(z)=\sup(\min(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y))/x.y=z)$

For $0<\alpha\le 0.5$

$$\mu_{\tilde{c}}(z)=\begin{cases}\mu^{L}{}_{\tilde{c}}(z)=\sup\left(\min\left(\dfrac{x-a_1}{2(a_2-a_1)},\dfrac{y-b_1}{2(b_2-b_1)}\right)/x.y=z\right) if\ a_1\le x\le a_2, b_1\le y\le b_2\\ \sup\left(\min\left(\dfrac{1}{2},\dfrac{1}{2}\right)/xy=z\right) if\ a_2\le x\le a_3, b_2\le y\le b_3, a_5\le x\le a_6, b_5\le y\le b_6\\ \mu^{R}{}_{\tilde{c}}(z)=\sup\left(\min\left(\dfrac{a_7-x}{2(a_7-a_6)},\dfrac{b_7-y}{2(b_7-b_6)}\right)/x.y=z\right) if\ a_6\le x\le a_7, b_6\le y\le b_7\\ 0,\text{otherwise}\end{cases}$$

If we choose $\alpha=\min\left(\dfrac{x-a_1}{2(a_2-a_1)},\dfrac{y-b_1}{2(b_2-b_1)}\right)$

Then $\alpha\le\dfrac{x-a_1}{2(a_2-a_1)},\alpha\le\dfrac{y-b_1}{2(b_2-b_1)}$ *hence* $\alpha^2\le\dfrac{x-a_1}{2(a_2-a_1)}\dfrac{y-b_1}{2(b_2-b_1)}$

$$xy-xb_1-ya_1+a_1b_1\ge 4\alpha^2(a_2-a_1)(b_2-b_1)$$

$$4\alpha^2(a_2-a_1)(b_2-b_1)+2\alpha\left(b_1(a_2-a_1)+a_1(b_2-b_1)\right)+a_1b_1-z\le 0$$

$$\alpha\le\frac{2a_1b_1-a_2b_1-a_1b_2\pm\sqrt{f_1(z)}}{4\left(a_1b_1-a_2b_1-a_1b_2+a_2b_2\right)}$$

Therefore $\mu^{L}{}_{\tilde{c}}(z)=\sup\alpha=\dfrac{2a_1b_1-a_2b_1-a_1b_2+\sqrt{f_1(z)}}{4\left(a_1b_1-a_2b_1-a_1b_2+a_2b_2\right)}$

Where $f_1(z)=\left(a_2b_1\right)^2-2a_1a_2b_1b_2+\left(a_1b_2\right)^2+4(a_2b_1-a_2b_1-a_1b_2+a_2b_2)z$

$$\frac{d\mu^{L}{}_{\tilde{c}}(z)}{dz}=\frac{f_1'(z)}{2\sqrt{f_1(z)}\left(a_1b_1-a_2b_1-a_1b_2+a_2b_2\right)}>0$$

Where $f_1'(z)=4(a_2b_1-a_2b_1-a_1b_2+a_2b_2)$

We note that $\dfrac{d\mu^{U}{}_{\tilde{c}}(z)}{dz}$ is both increases and decreases in (0, 0.5)

Similarly If we choose $\alpha=\min\left(\dfrac{a_7-x}{2(a_7-a_6)},\dfrac{b_7-y}{2(b_7-b_6)}\right)$

$$\mu^{U}_{\tilde{c}}(z) = \sup \alpha = \frac{2a_7b_7 - a_7b_6 - a_6b_7 - \sqrt{f_4(z)}}{4\left(a_6b_6 - a_7b_6 - a_6b_7 + a_7b_7\right)}$$

Where $f_4(z) = \left(a_7b_6\right)^2 - 2a_6a_7b_6b_7 + \left(a_6b_7\right)^2 + 4(a_6b_6 - a_7b_6 - a_6b_7 + a_7b_7)\,z$

Hence $\frac{d\mu^{U}_{\tilde{c}}(z)}{dz} < 0$ decreases in (0, 0.5)

For $0.5 < x, y < 1$

$$\mu_{\tilde{c}}(z) = \begin{cases} \mu^{L}_{\tilde{c}}(z) = \sup\left(\min\left(\frac{x-a_4}{2(a_4-a_3)}+1, \frac{y-b_4}{2(b_4-b_3)}+1\right) / \, xy = z\right) if\ a_3 \le x \le a_4, b_3 \le y \le b_4 \\ \mu^{R}_{\tilde{c}}(z) = \sup\left(\min\left(\frac{a_4-x}{2(a_5-a_4)}+1, \frac{b_4-y}{2(b_5-b_4)}+1\right) / \, xy = z\right) if\ a_4 \le x \le a_5, b_4 \le y \le b_5 \\ 0, \text{otherwise} \end{cases}$$

If we choose $\alpha = \min\left(\frac{x-a_4}{2(a_4-a_3)}+1, \frac{y-b_4}{2(b_4-b_3)}+1\right)$

$\alpha = \min\left(\frac{x-a_4}{2(a_4-a_3)}+1, \frac{y-b_4}{2(b_4-b_3)}+1\right)$ Then $\alpha \le \frac{x-a_4}{2(a_4-a_3)}+1,\ \alpha \le \frac{y-b_4}{2(b_4-b_3)}+1$

Also $\alpha^2 \le \left(\frac{x-a_4}{2(a_4-a_3)}+1\right)\left(\frac{y-b_4}{2(b_4-b_3)}+1\right)$

$$(x-a_4)(y-b_4) + 2(x-a_4)(b_4-b_3) + 2(y-b_4)(a_4-a_3) + 4(a_4-a_3)(b_4-b_3) \ge 4\alpha^2(a_4-a_3)(b_4-b_3)$$

Solving above equation we get

$$\alpha \le \frac{4a_3b_3 - 3a_4b_3 - 3a_3b_4 + 2a_4b_4 \pm \sqrt{f_2(z)}}{4\left(a_3b_3 - a_4b_3 - a_3b_4 + a_4b_4\right)}$$

Where $f_2(x) = \left(a_4b_3\right)^2 - 2a_3a_4b_3b_4 + \left(a_3b_4\right)^2 + 4(a_3b_3 - a_4b_3 - a_3b_4 + a_4b_4)\,z$

Therefore $\mu^{L}_{\tilde{c}}(z) = \sup \alpha = \frac{4a_3b_3 - 3a_4b_3 - 3a_3b_4 + 2a_4b_4 + \sqrt{f_2(z)}}{4\left(a_3b_3 - a_4b_3 - a_3b_4 + a_4b_4\right)}$

Similarly we can show that $\mu^{R}_{\tilde{c}}(z) = \sup \alpha = \frac{2a_4b_4 - 3a_5b_4 - 3a_4b_5 + 4a_5b_5 - \sqrt{f_3(x)}}{4\left(a_4b_4 - a_5b_4 - a_4b_5 + a_5b_5\right)}$

Where $f_3(x) = (a_5 b_4)^2 - 2a_4 a_5 b_4 b_5 + (a_4 b_5)^2 + 4(a_4 b_4 - a_5 b_4 - a_4 b_5 + a_5 b_5)x$

Hence we note that $\frac{d\mu^L_{\tilde{c}}(z)}{dz} > 0 \, and \, \frac{d\mu^U_{\tilde{c}}(z)}{dz} < 0$ increases and decreases respectively in (0.5, 1)

Hence the membership function is given by equation (3.8.1)

3.10 Product of two HpFNs using Co-ordinates method

We define $\tilde{A} \otimes \tilde{B} = \tilde{C} = \phi(\tilde{A}, \tilde{B})$

Where the ordinates are given by

$$x_1 = a_1 - 2\alpha(a_2 - a_1), y_1 = \mathrm{b}_1 - 2\alpha(\mathrm{b}_2 - b_1)$$

$$x_2 = a_7 - 2\alpha(a_7 - a_6), y_2 = \mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)$$

Now we form coordinates

$$c_1 = (x_1, y_1), c_2 = (x_1, y_2) \quad c_3 = (x_2, y_1), c_4 = (x_2, y_2)$$

$$\phi(c_1) = (x_1 y_1), \phi(c_2) = (x_1 y_2), \phi(c_3) = (x_2 y_1), \phi(c_4) = (x_2 y_2)$$

Where $\phi(c_2) < \phi(c_1) < \phi(c_4) < \phi(c_3)$

$$z = [\min \phi(c_1), \phi(c_2), \phi(c_3), \phi(c_4), \max \phi(c_1), \phi(c_2), \phi(c_3), \phi(c_4)]$$

$$z = [\phi(c_1), \phi(c_4)]$$

$$z = [\{a_1 + 2\alpha(a_2 - a_1)\}\{b_1 + 2\alpha(\mathrm{b}_2 - b_1)\}, \{a_7 - 2\alpha(a_7 - a_6)\}\{\mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)\}]$$

Now consider

$$x_3 = 2(\alpha - 1)(a_4 - a_3) + \mathrm{a}_4, y_3 = 2(\alpha - 1)(\mathrm{b}_4 - b_3) + b_4$$

$$x_4 = \mathrm{a}_4 - 2(\alpha - 1)(a_5 - a_4), y_4 = \mathrm{b}_4 - 2(\alpha - 1)(\mathrm{b}_5 - b_4)$$

$$c_5 = (x_3, y_3), c_6 = (x_3, y_4), c_7 = (x_4, y_3), c_8 = (x_4, y_4)$$

Now $\varphi(c_5) = (x_3 y_3), \varphi(c_6) = (x_3 y_4), \varphi(c_7) = (x_4 y_3), \varphi(c_8) = (x_4 y_4)$

$$z = [\min \varphi(c_5), \varphi(c_6), \varphi(c_7), \varphi(c_8), \max \varphi(c_5), \varphi(c_6), \varphi(c_7), \varphi(c_8)]$$

$$z = [\varphi(c_5), \varphi(c_8)]$$

$$z = [\{2(\alpha - 1)(a_4 - a_3) + \mathrm{a}_4\}\{2(\alpha - 1)(\mathrm{b}_4 - b_3) + b_4\}, \{\mathrm{a}_4 - 2(\alpha - 1)(a_5 - a_4)\}\{\mathrm{b}_4 - 2(\alpha - 1)(\mathrm{b}_5 - b_4)\}]$$

Hence we arrive product of two HpFN is another HpFNs.

3.11 Division of two HpFNs by $(\alpha)-$ cut method

For $0 \le \alpha \le 0.5$

$$A_\alpha (/) B_\alpha = \left[A_1^\alpha / B_2^\alpha , A_2^\alpha / B_1^\alpha \right]$$

$$= \left[\frac{a_1 + 2\alpha(a_2 - a_1)}{\mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)}, \frac{a_7 - 2\alpha(a_7 - a_6)}{b_1 + 2\alpha(\mathrm{b}_2 - b_1)} \right]$$

If $\mathrm{x} \in (a_1 / \mathrm{b}_7 , a_2 / \mathrm{b}_6)$, then $x = \frac{a_1 + 2\alpha(a_2 - a_1)}{\mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)}$

Thus $\alpha = \frac{a_1 - b_7 x}{2(a_1 - a_2 + b_6 x - b_7 x)}$

Similarly if $x \in \left[a_6 / b_2 , a_7 / b_1 \right]$ then $\mathrm{x} = \frac{a_7 - 2\alpha(a_7 - a_6)}{b_1 + 2\alpha(\mathrm{b}_2 - b_1)}$, we get $\alpha = \frac{a_7 - b_1 x}{-2(a_6 - a_7 + b_1 x - b_2 x)}$

For $0.5 \le \alpha \le 1$

$$A_\alpha (/) B_\alpha = \left[A_1^\alpha / B_2^\alpha , A_2^\alpha / B_1^\alpha \right]$$

$$= \left[\frac{2(\alpha - 1)(a_4 - a_3) + \mathrm{a}_4}{\mathrm{b}_4 - 2(\alpha - 1)(\mathrm{b}_5 - b_4)}, \frac{\mathrm{a}_4 - 2(\alpha - 1)(a_5 - a_4)}{2(\alpha - 1)(\mathrm{b}_4 - b_3) + b_4} \right]$$

If $x \in \left[a_3 / b_5 , a_4 / b_4 \right]$ then $x = \frac{2(\alpha - 1)(a_4 - a_3) + \mathrm{a}_4}{\mathrm{b}_4 - 2(\alpha - 1)(\mathrm{b}_5 - b_4)}$

Thus $\alpha = \frac{2a_3 - a_4 + b_4 x - 2b_5 x}{2(a_3 - a_4 + b_4 x - b_5 x)}$

Similarly if $x \in \left[a_4 / b_4 , a_5 / b_3 \right]$ then $x = \frac{\mathrm{a}_4 - 2(\alpha - 1)(a_5 - a_4)}{2(\alpha - 1)(\mathrm{b}_4 - b_3) + b_4}$

We get $\alpha = \frac{a_4 - 2a_5 + b_3 x - b_4 x}{2(a_4 - a_5 + b_3 x - b_4 x)}$

Therefore

$$\mu_{\tilde{A}_{Hp}(/)\tilde{B}_{Hp}}(x)=\begin{cases}\dfrac{a_1-b_7x}{2(a_1-a_2+b_6x-b_7x)}, & a_1/b_7\le x\le a_2/b_6\\ \dfrac{1}{2}, & a_2/b_6\le x<a_3/b_5\\ \dfrac{2a_3-a_4+b_4x-2b_5x}{2(a_3-a_4+b_4x-b_5x)}, & a_3/b_5\le x\le a_4/b_4\\ \dfrac{a_4-2a_5+2b_3x-b_4x}{2(a_4-a_5+b_3x-b_4x)}. & a_4/b_4\le x\le a_5/b_3\\ \dfrac{1}{2}, & a_5/b_3\le x\le a_6/b_2\\ \dfrac{a_7-b_1x}{-2(a_6-a_7+b_1x-b_2x)}, & a_6/b_2\le x\le a_7/b_1\\ 0, & otherwise\end{cases} \tag{3.11.1}$$

Thus If $\tilde{A}_{Hp}=(a_1,a_2,a_3,\mathrm{a}_4,\mathrm{a}_5,\mathrm{a}_6,\mathrm{a}_7)$ *and* $\tilde{B}_{Hp}=(\mathrm{b}_1,\mathrm{b}_2,\mathrm{b}_3,\mathrm{b}_4,\mathrm{b}_5,\mathrm{b}_6,\mathrm{b}_7)$ are any two HpFS then $\tilde{A}_{Hp}(/)\tilde{B}_{Hp}$ is also HpFS and it is given by $\tilde{A}_{Hp}(/)\tilde{B}_{Hp}=(a_1/b_7,a_2/b_6,a_3/b_5,a_4/b_4,a_5/b_3,a_6/b_2,a_7/b_1)$

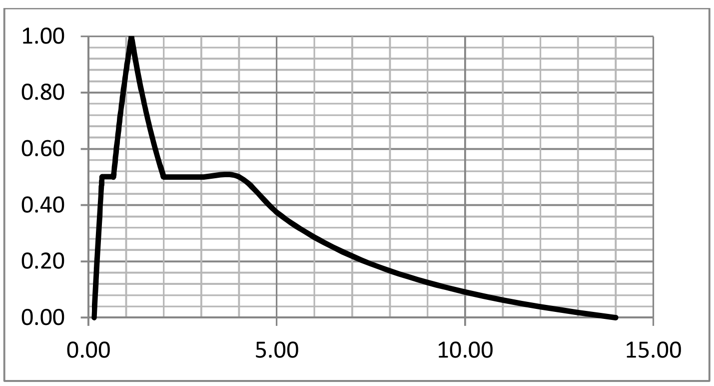

Fig. 8. Graphical representation of the membership function of division of two the HpFNs

3.12 Division of two HpFN by extension principle method

Let $\tilde{A}(/)\tilde{B}=\tilde{C}$ where $\mu_{\tilde{C}}(z)=\sup(\min(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(\mathrm{y}))/(x/y)=z)$

For $0<\alpha\le 0.5$

$$\mu_{\tilde{C}}(z)=\begin{cases}\mu^{L}{}_{\tilde{C}}(z)=\sup\left(\min\left(\dfrac{x-a_1}{2(a_2-a_1)},\dfrac{y-b_7}{2(b_6-b_7)}\right)/\left(\dfrac{x}{y}\right)=z\right) if\ a_1\le x\le a_2, b_6\le y\le b_7\\ \sup\left(\min\left(\dfrac{1}{2},\dfrac{1}{2}\right)/\left(\dfrac{x}{y}\right)=z\right) if\ a_2\le x\le a_3, b_2\le y\le b_3, a_5\le x\le a_6, b_5\le y\le b_6\\ \mu^{R}{}_{\tilde{C}}(z)=\sup\left(\min\left(\dfrac{a_7-x}{2(a_7-a_6)},\dfrac{x-b_1}{2(\mathrm{b}_2-b_1)}\right)/\left(\dfrac{x}{y}\right)=z\right) if\ a_6\le x\le a_7, b_1\le y\le b_2\\ 0, \text{otherwise}\end{cases}$$

If we choose $\min\left(\frac{x-a_1}{2(a_2-a_1)},\frac{y-b_7}{2(b_6-b_7)}\right)=\alpha$

$$\alpha\le\frac{x-a_1}{2(a_2-a_1)},\alpha\le\frac{\frac{x}{z}-b_7}{2(b_6-b_7)}\ ,\sin ce\left(\frac{x}{y}\right)=z$$

$$therefore\ \frac{a_1-b_7z}{2(a_1-a_7+b_6z-b_7\ \mathrm{z})}\ge\alpha$$

$$\mu^L{}_{\tilde{c}}(z)=\sup\alpha=\frac{a_1-b_7z}{2(a_1-a_7+b_6z-b_7\ \mathrm{z})}$$

Similarly $\mu^R{}_{\tilde{c}}(z)=\sup\alpha=\frac{a_1-b_7z}{2(a_1-a_7+b_6z-b_7\ \mathrm{z})}$

For $0.5<\alpha\le1$

$$\mu_{\tilde{c}}(z)=\begin{cases}\mu^L{}_{\tilde{c}}(z)=\sup\left(\min\left(\frac{x-a_4}{2(a_4-a_3)}+1,\frac{b_4-y}{2(b_5-b_4)}+1\right)/\,x-y=z\right) if\ a_3\le x\le a_4,b_4\le y\le b_5\\ \frac{1}{2} if\ a_2\le x\le a_3,b_2\le y\le b_3,a_5\le x\le a_6,b_5\le y\le b_6\\ \mu^R{}_{\tilde{c}}(z)=\sup\left(\min\left(\frac{a_4-x}{2(a_5-a_4)}+1,\frac{y-b_4}{2(b_4-b_3)}+1\right)/\,x-y=z\right) if\ a_4\le x\le a_5,b_3\le y\le b_4\\ 0,\text{otherwise}\end{cases}$$

By using above procedure we find

$$\mu^L{}_{\tilde{c}}(z)=\sup\alpha=\frac{2a_3-a_4+b_4z-2b_5z}{2(a_3-a_4+b_4z-b_5\ \mathrm{z})}$$

$$\mu^R{}_{\tilde{c}}(z)=\sup\alpha=\frac{a_4-2a_5+b_3z-b_4z}{2(a_4-a_5+b_3z-b_4\ \mathrm{z})}$$

Hence the membership function is given by equation (3.11.1)

3.13 Division of two HpFN by coordinate method

We define $\tilde{A}(/)\tilde{B}=\tilde{C}=\phi(\tilde{A},\tilde{B})$

The ordinates are given by

$$x_1 = a_1 - 2\alpha(a_2 - a_1), y_1 = \mathrm{b}_1 - 2\alpha(\mathrm{b}_2 - b_1)$$

$$x_2 = a_7 - 2\alpha(a_7 - a_6), y_2 = \mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)$$

Now we form co-ordinates

$$c_1 = (x_1, y_1), c_2 = (x_1, y_2) \quad c_3 = (x_2, y_1), c_4 = (x_2, y_2)$$

$$\phi(c_1) = (x_1 / y_1), \phi(c_2) = (x_1 / y_2), \phi(c_3) = (x_2 / y_1), \phi(c_4) = (x_2 / y_2)$$

$$z = [\min \phi(c_1), \phi(c_2), \phi(c_3), \phi(c_4), \max \phi(c_1), \phi(c_2), \phi(c_3), \phi(c_4)]$$

$$z = [\phi(c_2), \phi(c_3)]$$

$$z = \left[\frac{a_1 + 2\alpha(a_2 - a_1)}{\mathrm{b}_7 - 2\alpha(\mathrm{b}_7 - b_6)}, \frac{a_7 - 2\alpha(a_7 - a_6)}{b_1 + 2\alpha(\mathrm{b}_2 - b_1)} \right]$$

Similarly we can find that

$$z = \left[\frac{2(\alpha - 1)(a_4 - a_3) + \mathrm{a}_4}{\mathrm{b}_4 - 2(\alpha - 1)(\mathrm{b}_5 - b_4)}, \frac{\mathrm{a}_4 - 2(\alpha - 1)(a_5 - a_4)}{2(\alpha - 1)(\mathrm{b}_4 - b_3) + b_4} \right]$$

Hence the membership function is given by equation (3.11.1).

4 Conclusion

In this paper, we have introduced Heptagon fuzzy numbers which deals with the membership function of seven numbers, the four arithmetic operations based on α-cuts, extension principle, and co-ordinate method were discussed and corresponding membership functions are obtained .The membership function obtained by these methods are same which validate our results The above concepts can be extended to intuitionistic and generalized fuzzy numbers. Further HpFNs can be applied in various engineering and mathematical sciences.

Competing Interests

Author has declared that no competing interests exist.

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